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A New Approach to the Relativistic Schrödinger Equation with Central Potential: Ansatz Method

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Applying an ansatz to the eigenfunction, we obtain the exact closed-form solutions of the relativistic Schrödinger equation with the potential $V(r) = -a/r + b/r^{1/2}$ both in three dimensions and in two dimensions. The restrictions on the parameters of the given potential and the angular momentum quantum number are also presented.

1. INTRODUCTION

The exact solutions of fundamental dynamical equations play an important role in physics. It is possible to obtain the exact solution of the Schrödinger equation with central physical potentials by applying an ansatz to the eigenfunction and restricting the parameters of the given potential and the angular momentum quantum number [6, 12–15, 22–24, 31, 34, 38, 39]. During the past several decades, much effort has gone into studying the stationary Schrödinger equation with central potentials containing negative powers of the radial coordinate [1–5, 7–11, 16–21, 25–30, 32, 33, 35–37]. Interest in these central physical potentials stems from the fact that the study of the relevant Schrödinger equation provides insight into the physical problem. Most studies have been carried out in the nonrelativistic case. The study of the Schrödinger equation with a physical potential in the relativistic case is beyond our scope. The purpose of this paper is to study the relativistic

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Schrödinger equation with a given potential and then generalize it to the two-dimensional case, in accord with recent interest in lower dimensional field theory. This study is based on our previous work [12–15].

This paper is organized as follows. In Section 2, applying an ansatz to the eigenfunction, we obtain the solution of the relativistic Schrödinger equation with this potential in three dimensions. The two-dimensional case is presented in Section 3. Concluding remarks are given in Section 4.

2. THREE-DIMENSIONAL CASE

Natural units $\hbar = c = 1$ are employed throughout this paper if not explicitly stated otherwise. Consider the relativistic Schrödinger equation

$$(-\nabla^2 + M^2)\psi(r) = (E - V(r))^2\psi(r) \tag{1}$$

with the potential

$$V(r) = -\frac{a}{r} + \frac{b}{\sqrt{r}} \tag{2}$$

where M and E denote the mass and energy, respectively. Let

$$\psi(r, \theta, \varphi) = R_l(r)Y_{lm}(\theta, \varphi) \tag{3}$$

where l denotes the angular momentum quantum number. On substituting Eq. (3) into Eq. (1), we find that $R_l(r)$ satisfies

$$\frac{d^2R_l(r)}{dr^2} + \frac{2}{r}\frac{dR_l(r)}{dr} + \left[(E - V(r))^2 - M^2 - \frac{l(l+1)}{r^2}\right]R_l(r) = 0 \tag{4}$$

For the solution of Eq. (4), applying an ansatz to the radial wave function

$$R_l(r) = \exp[p(r)] \sum_{n=0} a_n r^{n/2+\nu} \tag{5}$$

where

$$p(r) = \alpha r + 2\beta r^{1/2} \tag{6}$$

Substituting Eq. (5) into Eq. (4), we obtain the following recursion relation by setting the coefficient of $r^{n/2+\nu-1}$ to zero:

$$A_n a_n + B_{n+1}a_{n+1} + C_{n+2}a_{n+2} = 0 \tag{7}$$

where

$$A_n = \beta^2 + b^2 + 2aE + \alpha(n + 2\nu + 2) \tag{8a}$$

$$B_n = -2ab + \beta(3/2 + n + 2\nu) \tag{8b}$$

$$C_n = a^2 + (\nu + n/2)(1 + \nu + n/2) - l(l + 1) \tag{8c}$$

and

$$\alpha^2 = M^2 - E^2 \tag{9a}$$

$$\alpha\beta = bE \tag{9b}$$

It is easy to obtain from Eq. (9a)

$$\alpha = \pm\sqrt{M^2 - E^2} \tag{10}$$

In order to retain the well-behaved solution at the origin and at infinity, we choose α as

$$\alpha = -\sqrt{M^2 - E^2} \tag{11}$$

from which, together with Eq. (9b), we have

$$\beta = -\frac{bE}{\sqrt{M^2 - E^2}} \tag{12}$$

Similar to the work refs. 6, 14, and 15, if the first nonvanishing coefficient is $a_0 \neq 0$ in Eq. (7), we can obtain $C_0 = 0$ from Eq. (8c), i.e.,

$$\nu_\pm = -\frac{1}{2} \pm \sqrt{(l + 1/2)^2 - a^2} \tag{13}$$

With the same reasoning, we choose ν_+ as a physically acceptable solution, i.e.,

$$\nu_+ = -1/2 + \xi$$

where

$$\xi \equiv \sqrt{(l + 1/2)^2 - a^2}$$

Furthermore, if the pth nonvanishing coefficient $a_p \neq 0$, but $a_{p+1} = a_{p+2} = a_{p+3} = \cdots = 0$, then it can be shown from Eq. (8a) that $A_p = 0$, i.e.,

$$b^2M^2 + 2aE(M^2 - E^2) = (p + 2 + 2\nu_+)(M^2 - E^2)\sqrt{M^2 - E^2} \tag{14}$$

from which we can obtain the corresponding energy eigenvalue. As we know, A_n, B_n, and C_n must satisfy the determinant relation for a nontrivial solution,

$$\det\begin{vmatrix} B_0 & C_1 & \cdots & \cdots & \cdots & 0 \\ A_0 & B_1 & C_2 & & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{p-1} & B_p \end{vmatrix} = 0 \tag{15}$$

In order to appreciate this method, we present the exact solutions for the cases $p = 0, 1$ as follows.

1. When $p = 0$, we have

$$b^2M^2 + 2aE_0(M^2 - E_0^2) = 2(1 + \nu_+)(M^2 - E_0^2)\sqrt{M^2 - E_0^2} \qquad (16)$$

from Eq. (14). Obviously, the E_0 can be evaluated by Eq. (16) if the values of the parameters of the potential and angular momentum quantum number are given. Moreover, we can obtain the restriction on the parameters of the potential and the angular momentum quantum number from Eq. (15), that is, we can obtain $B_0 = 0$, which, together with Eqs. (11)–(13), leads to

$$E_0(4\nu_+ + 3) + 4a\sqrt{M^2 - E_0^2} = 0 \qquad (17)$$

The corresponding eigenfunction for $p = 0$ can now be given as

$$R_\ell^{(0)} = a_0 r^{\nu_+} \exp\left[-\sqrt{M^2 - E_0^2}\, r - \frac{2bE_0}{\sqrt{M^2 - E_0^2}}\, r^{1/2}\right] \qquad (18)$$

where a_0 is the normalization constant.

2. When $p = 1$, it is found from Eq. (14) that

$$b^2M^2 + 2aE_1(M^2 - E_1^2) = (3 + 2\nu_+)(M^2 - E_1^2)\sqrt{M^2 - E_1^2} \qquad (19)$$

Similarly, one can obtain the energy eigenvalue E_1 from Eq. (19). The corresponding restriction on the parameters of the potential and angular momentum quantum number can also be found from Eq. (15), i.e.,

$$B_0B_1 - A_0C_1 = 0 \qquad (20)$$

which enables us to write

$$\frac{b^2(E_1 + 4E_1\xi - 4a\eta)((3 + 4E_1\xi - 4a\eta) - (1 + 4\xi)\{b^2M^2 - \eta^2[-2aE_1 + (1 + 2\xi)\eta]\})}{4\eta^2} = 0 \qquad (21)$$

where

$$\eta \equiv \sqrt{M^2 - E_1^2}$$

The corresponding eigenfunction for $p = 1$ can be written as

$$R_\ell^{(1)} = (a_0 + a_1 r^{1/2}) r^{\nu_+} \exp\left[-\sqrt{M^2 - E_1^2}\, r - \frac{2bE_1}{\sqrt{M^2 - E_1^2}}\, r^{1/2}\right] \qquad (22)$$

where a_i ($i = 0, 1$) can be evaluated by the normalization condition.

Following in this way, we can generate a class of exact solutions by setting $p = 1, 2, \ldots$. Generally, if the pth nonvanishing coefficient is $a_p \neq$

0, but $a_{p+1} = a_{p+2} = \cdots = 0$, then we can obtain the energy eigenvalue E_p from Eq. (14). The corresponding eigenfunction is

$$R_\ell^{(p)} = (a_0 + a_1 r^{1/2} + \cdots + a_p r^{p/2}) r^{\nu_+} \exp\left[-\sqrt{M^2 - E_p^2}\, r - \frac{2bE_p}{\sqrt{M^2 - E_p^2}}\, r^{1/2}\right] \tag{23}$$

where a_i ($i = 1, 2, \ldots, p$) can be expressed by Eq. (7) and in principle obtained by the normalization condition.

3. TWO-DIMENSIONAL CASE

Let us now turn to the two-dimensional relativistic Schrödinger equation. In this case, we can take the wave function as

$$\psi(\mathbf{r}, \varphi) = R_m(r) e^{\pm im\varphi}, \qquad m = 0, 1, 2, \ldots \tag{24}$$

from which we obtain

$$\frac{d^2 R_m(r)}{dr^2} + \frac{1}{r}\frac{dR_m(r)}{dr} + \left[(E - V(r))^2 - M^2 - \frac{m^2 - 1/4}{r^2}\right] R_m(r) = 0 \tag{25}$$

where M, m, and E denote the mass, angular momentum quantum number, and energy, respectively. For the solution of Eq. (25), similarly, we apply the following ansatz to the radial wave function $R_m(r)$:

$$R_m(r) = \exp[p(r)] \sum_{n=0} a_n r^{n/2+\nu} \tag{26}$$

where

$$p(r) = \alpha_1 r + 2\beta_1 r^{1/2} \tag{27}$$

Substituting Eq. (26) into Eq. (25), we obtain the following recursion relation:

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \tag{28}$$

where

$$A_n = \beta_1^2 + b^2 + 2aE + \alpha_1(n + 2\nu + 1) \tag{29a}$$

$$B_n = -2ab + \beta_1(1/2 + n + 2\nu) \tag{29b}$$

$$C_n = a^2 + (\nu + n/2)(\nu + n/2) - (m^2 - 1/4) \tag{29c}$$

and

$$\alpha_1^2 = M^2 - E^2 \tag{30a}$$

$$\alpha_1 \beta_1 = bE \tag{30b}$$

Similarly, the physically acceptable solutions α_1 and β_1 can be obtained from Eq. (30),

$$\alpha_1 = -\sqrt{M^2 - E^2} \tag{31}$$

$$\beta_1 = -\frac{bE}{\sqrt{M^2 - E^2}} \tag{32}$$

Furthermore, if the first nonvanishing coefficient is $a_0 \neq 0$ in Eq. (28), we then obtain $C_0 = 0$ from Eq. (29c), i.e.,

$$\nu_{\pm} = \pm\sqrt{m^2 - a^2 - 1/4} \tag{33}$$

Likewise, we choose ν_+ as a physically acceptable solution. If the pth nonvanishing coefficient is $a_p \neq 0$, but $a_{p+1} = a_{p+2} = a_{p+3} = \cdots = 0$, then $A_p = 0$, from Eq. (29a), i.e.,

$$b^2M^2 + 2aE(M^2 - E^2) = (p + 1 + 2\nu_+)(M^2 - E^2)\sqrt{M^2 - E^2} \tag{34}$$

from which we can obtain the energy eigenvalue. Certainly, A_n, B_n, and C_n must also satisfy Eq. (15) for a nontrivial solution. The exact solutions of the different cases $p = 0, 1$ are given below to demonstrate the method.

1. When $p = 0$, we have

$$b^2M^2 + 2aE_0(M^2 - E_0^2) = (2\nu_+ + 1)(M^2 - E_0^2)\sqrt{M^2 - E_0^2} \tag{35}$$

from Eq. (34). Clearly, the E_0 can be evaluated from Eq. (35) if the values of the parameters and angular momentum quantum number are given. In addition, the restriction on the parameters of the potential and the angular momentum quantum number can be obtained from Eq. (15), namely, we can obtain $B_0 = 0$, which, together with Eqs. (31)–(32), leads to

$$E_0(1 + 4\nu_+) = 4a\sqrt{M^2 - E_0^2} \tag{36}$$

The corresponding eigenfunction for $p = 0$ can now be written as

$$R_m^{(0)} = a_0 r^{\nu_+} \exp\left[-\sqrt{M^2 - E_0^2}\,r - \frac{2bE_0}{\sqrt{M^2 - E_0^2}}\,r^{1/2}\right] \tag{37}$$

where a_0 is the normalization constant.

2. When $p = 1$, it can be shown from Eq. (34) that

$$b^2M^2 + 2aE_1(M^2 - E_1^2) = 2(1 + \nu_+)(M^2 - E_1^2)\sqrt{M^2 - E_1^2} \tag{38}$$

from which we can obtain the energy eigenvalue. The corresponding restriction can also be obtained from Eq. (15), i.e.,

$$B_0B_1 - A_0C_1 = 0 \tag{39}$$

which leads to

$$\left(2ab - \frac{b(1/2 + 2\nu_+)E_1}{\eta}\right)\left(2ab - \frac{b(3/2 + 2\nu_+)E_1}{\eta}\right)$$
$$= \frac{(1 + 4\nu_+)\{b^2 M^2 - \eta^2[-2aE_1 + (1 + 2\nu_+)\eta]\}}{4\eta^2} \tag{40}$$

where ν_+ is given in Eq. (33) and η is also defined above. The corresponding eigenfunction for $p = 1$ is

$$R_m^{(1)} = (a_0 + a_1 r^{1/2}) r^{\nu_+} \exp\left[-\sqrt{M^2 - E_1^2}\, r - \frac{2bE_1}{\sqrt{M^2 - E_1^2}}\, r^{1/2}\right] \tag{41}$$

where a_i ($i = 0, 1$) can be evaluated by the normalization condition.

Similar to the three-dimensional case, we can generate a class of exact solutions by setting $p = 1, 2, \ldots,$ Generally, if the pth nonvanishing coefficient is $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, then we can obtain the energy eigenvalue from Eq. (34). The corresponding eigenfunction can be written as

$$R_m^{(p)} = (a_0 + a_1 r^{1/2} + \cdots + a_p r^{p/2}) r^{\nu_+} \exp\left[-\sqrt{M^2 - E_p^2}\, r - \frac{2bE_p}{\sqrt{M^2 - E_p^2}}\, r^{1/2}\right] \tag{42}$$

where a_i ($i = 1, 2, \ldots, p$) can be expressed by Eq. (28) and be obtained by the normalization condition.

4. CONCLUDING REMARKS

In this paper, applying an ansatz to the eigenfunction, we first obtained the solutions of the relativistic Schrödinger equation with the given potential in three dimensions. Due to wide interest in lower dimensional field theory, we then generalized this problem to the two-dimensional case. The corresponding restrictions on the parameters of the given potential and the angular momentum quantum number were presented. Study of the relativistic Schrödinger equation with other central potentials is in progress.

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REFERENCES

1. D. Emin and T. Holstein (1976). *Phys. Rev. Lett.* **36**, 323.
2. D. Emin (1982). *Phys. Today* **35**, 35.
3. A. O. Barut (1980). *J. Math. Phys.* **21**, 568.
4. D. R. Bates and I. Esterman (1970). *Advances in Atomic and Molecular Physics*, Vol. 6 (Academic Press, New York).
5. B. H. Bransden and C. J. Joachain (1983). *Physics of Atoms and Molecules* (Longman, London).
6. S. K. Bose and N. Gupta (1998). *Nuovo Cimento* **113B**, 299.
7. M. S. Child (1991). *Semiclassical Mechanics with Molecular Applications* (Clarendon Press, Oxford).
8. F. Calogero (1967). *Variable Phase Approach to Potential Scattering* (Academic, Press, New York).
9. S. Coleman (1988). *Aspects of Symmetry* (Cambridge University Press, Cambridge), p. 234.
10. A. De Alfaro and T. Regge (1965). *Potential Scattering* (North-Holland, Amsterdam).
11. A. De Souza Dutra (1988). *Phys. Lett. A* **131**, 319.
12. S. H. Dong and Z. Q. Ma (1998). *J. Phys. A* **31**, 9855.
13. S. H. Dong, Z. Q. Ma, and G. Esposito (1999). *Found Phys. Lett.* **12**, 465.
14. S. H. Dong (2000). *Int. J. Theor. Phys.* **39**, 1119.
15. S. H. Dong, M. S. Child, and X. G. Wang (2000). *J. Phys. A.* **33**, 5653.
16. G. Esposito (1998a). *J. Phys. A* **31**, 9493.
17. G. Esposito (1998b). *Found. Phys. Lett.* **11**, 535.
18. C. Farina De Souza and A. De Souza Dutra (1987). *Phys. Lett. A* **123**, 297.
19. H. Haken. *Encyclopedia of Physics*, Vol. **25/20** (Van Nostrand, Princeton, New Jersey).
20. H. L. Hall (1985). *Phys. Rev. A* **32**, 14.
21. H. L. Hall (1986). *J. Phys. A* **12**, 2079.
22. R. S. Kaushal (1991). *Ann. Phys.* (N.Y.) **206**, 90.
23. R. S. Kaushal and D. Parashar (1992). *Phys. Lett. A* **170**, 335.
24. R. S. Kaushal (1989). *Phys. Lett. A* **142**, 57.
25. L. D. Landau and E. M. Lifshitz (1977). *Quantum Mechanics*, Vol. 3, 3rd ed. (Pergamon Press, Oxford).
26. R. J. LeRoy and W. Lam (1970). *Chem. Phys. Lett.* **71**, 544.
27. R. J. LeRoy and R. B. Bernstein (1970). *J. Chem. Phys.* **52**, 3869.
28. G. C. Maitland, M. M. Higby, E. B. Smith, and V. A. Wahoham (1987). *Intermolecular Forces* (Oxford University Press, Oxford).
29. C. L. Nehta and S. H. Patil (1978). *Phys. Rev. A* **17**, 34.
30. R. G. Newton (1967) *Scattering Theory of Waves and Particles* (McGraw-Hill, New York).
31. S. Özcelik and M. Simsek (1991). *Phys. Lett. A* **152**, 145.
32. A. Salam and J. Strathdee (1976). *Phys. Rev. D* **1**, 3296.
33. A. Khare and S. N. Behra (1980). *Pramana J. Phys.* **4**, 327.

34. M. Simsek and S. Özcelik (1994). *Phys. Lett. A* **186**, 35.
35. H. Stillinger (1979). *J. Math. Phys.* **20**, 1891.
36. E. Vogt and G. H. Wannier (1954). *Phys. Rev.* **95**, 1190.
37. A. Voros (1999). *J. Phys. A* **32**, 5993.
38. M. Znojil (1989). *J. Math. Phys.* **30**, 23.
39. M. Znojil (1990). *J. Math. Phys.* **31**, 108.